PICTURE OF THE MONTH

This TIROS V photograph shows a remarkable large-scale band of up-slope stratus and frontal cloudiness just east of the Rocky Mountains. The photograph was taken on December 11, 1962, at 1832 GMT (pass 2512, camera 1, frame 8) and was received at Point Mugu, Calif. via direct readout. The center-cross fiducial mark is located approximately 80 mi. northeast of Albuquerque, N. Mex. near the crest of the Rockies. North is toward the top of the picture.

At the time of this photograph a recent surge of Arctic air had invaded the Great Plains. Midday surface temperatures over Kansas were in the teens, whereas over the western portions of Wyoming, Colorado, and New Mexico they were in the 30's and low 40's. The quasi-stationary front separating the two air masses lay north-south along the eastern slope of the Rockies, nearly coincident with the well-defined western edge of the cloud band. At the western edge, the cloudiness was low stratiform, lifting and thinning out eastward, and becoming broken middle and upper layers over Kansas and Oklahoma (northeastern quadrant of photograph).

The snow-covered higher elevations of the Colorado Rockies appear north and northwest of the center-cross fiducial mark. However, skies in that area were not completely clear; ground observers were reporting variable amounts of thin cirrus, largely invisible in this photograph. Thicker cirrus does appear toward the southwest corner.

The slightly inferior quality of the lower half of the picture is due to electronic "noise".

COMPARISON OF ANALYTICAL AND NUMERICAL SOLUTIONS TO AN INITIAL VALUE PROBLEM DEFINED BY A LINEARIZED, QUASI-GEOSTROPHIC MODEL

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ABSTRACT

The accuracy of a numerical technique devised for the purpose of obtaining approximate solutions to an initial value problem defined by linearized equations for quasi-geostrophic flow is tested in certain simple cases for which it is possible to obtain closed solutions. The numerical technique is found to be extremely accurate.

1. INTRODUCTION

Numerical solutions to initial value problems defined by linearized equations for quasi-geostrophic flow provide useful information concerning certain meteorological problems (for examples, see [1, 2]). Information concerning the accuracy of the numerical techniques used to obtain such solutions would, therefore, seem to be of general interest.

Before proceeding with the computations described in [1], the author tested the numerical technique used there by applying it to certain simple cases for which it was possible to obtain closed solutions. The present paper has been prepared to make the results of these tests available.

2. BASIC EQUATIONS

The model is defined by,

$$\mathbf{V}=\frac{\mathbf{k}}{f_0}\times\nabla\phi \tag{1}$$

$$\frac{\partial\zeta}{\partial t}+\mathbf{V}\cdot\nabla(f+\zeta)=f_0\frac{\partial\omega}{\partial p} \tag{2}$$

and

$$\frac{\partial^2\phi}{\partial p\partial t}+\mathbf{V}\cdot\nabla\frac{\partial\phi}{\partial p}+\sigma\omega=0 \tag{3}$$

$\mathbf{V}$ is the horizontal wind, $\mathbf{k}$ is a unit-vertical vector, ϕ is the geopotential, $\zeta=f_0^{-1}\nabla^2\phi$ is the relative vorticity, f is the Coriolis parameter, f_0 is a standard value of f, ω is the individual derivative of pressure, p is pressure, and

$$\sigma=\theta^{-1}\frac{\partial\phi}{\partial p}\frac{\partial\theta}{\partial p} \tag{4}$$

which, at most, is taken to be a function of pressure alone.

The dependent variables are written

$$\phi=\bar{\phi}(p,y)+\phi'(x,y,p,t), \tag{5}$$

$$\mathbf{V}=U(p)\mathbf{i}+\mathbf{V}'(x,y,p,t) \tag{6}$$

and

$$\omega=\omega'(x,y,p,t) \tag{7}$$

The primes denote perturbation quantities. $\bar{\phi}\ (p,y)$ is the geopotential of the base state, $U\ (p)$ is the mean zonal wind, $\mathbf{i}$ is a unit vector pointing eastward, x is east-west distance and y is north-south distance. From the geostrophic relationship,

$$U(p)=-f_0^{-1}\frac{\partial\bar{\phi}}{\partial y} \tag{8}$$

By the usual technique, we linearize equations (2) and (3) to obtain

$$\nabla^2\frac{\partial\phi'}{\partial t}+U\nabla^2\frac{\partial\phi'}{\partial x}+\beta\frac{\partial\phi'}{\partial x}=f_0^2\frac{\partial\omega'}{\partial p} \tag{9}$$

$$\frac{\partial^2\phi'}{\partial p\partial t}+U\frac{\partial^2\phi'}{\partial p\partial x}-\frac{dU}{dp}\frac{\partial\phi'}{\partial x}+\sigma\omega'=0 \tag{10}$$

Elimination of the time derivatives between equations (9) and (10) yields

$$\sigma\nabla^2\omega'+f_0^2\frac{\partial^2\omega'}{\partial p^2}=2\frac{dU}{dp}\nabla^2\frac{\partial\phi'}{\partial x}+\beta\frac{\partial^2\phi'}{\partial p\partial x} \tag{11}$$

We assume solutions of the form

$$z'=\frac{\phi'}{g}=A(p,t)\sin kx+B(p,t)\cos kx \tag{12}$$

$$\omega'=C(p,t)\sin kx+D(p,t)\cos kx \tag{13}$$

Alternately, we may write

$$z'=R_z \cos (kx+\delta_z) \tag{14}$$

$$\omega'=R_\omega \cos (kx+\delta_\omega) \tag{15}$$

where

$$R_z=(A^2+B^2)^{1/2} \tag{16}$$

$$R_\omega=(C^2+D^2)^{1/2} \tag{17}$$

$$\tan \delta_z=A/B \tag{18}$$

and

$$\tan \delta_\omega=C/D \tag{19}$$

When equations (12) and (13) are substituted into (9) and (11) and coefficients of sin kx and cos kx are equated, one obtains

$$\frac{\partial A}{\partial t}=kC_R B-g^{-1}\left(\frac{f_0}{k}\right)^2 \frac{\partial C}{\partial p} \tag{20}$$

$$\frac{\partial B}{\partial t}=-kC_R A-g^{-1}\left(\frac{f_0}{k}\right)^2 \frac{\partial D}{\partial p} \tag{21}$$

$$\frac{\partial^2 C}{\partial p^2}-\sigma\left(\frac{k}{f_0}\right)^2 C=2kg\left(\frac{k}{f_0}\right)^2 \frac{\partial U}{dp} B-\frac{\beta kg}{f_0^2}\frac{\partial B}{\partial p} \tag{22}$$

and

$$\frac{\partial^2 D}{\partial p^2}-\sigma\left(\frac{k}{f_0}\right)^2 D=-2kg\left(\frac{k}{f_0}\right)^2 \frac{dU}{dp} A+\frac{\beta kg}{f_0^2}\frac{\partial A}{\partial p} \tag{23}$$

which are essentially the same as obtained by Wiin-Nielsen [2]. Equations (20) to (23) differ from those employed in [1] only to the extent that a harmonic dependence on the meridional coordinate was allowed in the previous study.

The quantity, C_R, is the Rossby wave speed.

$$C_R=U-(\beta/k^2) \tag{24}$$

The system of equations (20–23) may be solved numerically as an initial value problem through a simple computational cycle. Given initial values of A and B, initial values of C and D are obtained by solution of the diagnostic equations (22) and (23). Equations (20) and (21) may then be used to obtain A and B at the next time step. The cycle may then be repeated until the required time interval has been spanned.

3. NUMERICAL TECHNIQUES

The finite-difference grid to be applied to the vertical coordinate is shown by table 1. A high degree of vertical resolution was required to portray the vertical structure of the disturbances treated in the author's previous paper [1]. It is possible that the relatively simple disturbances treated in the present paper would allow the use of a substantially coarser mesh. However, since the purpose of the computations reported on here was merely to test the computational procedure employed in [1], calculations were performed only with the fine grid.

By use of the subscript, i, to denote properties at the ith grid point, the diagnostic equations (22) and (23), are approximated by

Table 1.—*Description of finite difference grid*

Grid-Point Index	Grid-Point Pressure (mb.)	Equation Applied
1	0	
2	31.25	vorticity
3	62.50	omega
4	93.75	vorticity
5	125.00	omega
6	156.25	vorticity
7	187.50	omega
8	218.75	vorticity
9	250.00	omega
10	281.25	vorticity
11	312.50	omega
12	343.75	vorticity
13	375.00	omega
14	406.25	vorticity
15	437.50	omega
16	468.75	vorticity
17	500.00	omega
18	531.25	vorticity
19	562.50	omega
20	593.75	vorticity
21	625.00	omega
22	656.25	vorticity
23	687.50	omega
24	718.75	vorticity
25	750.00	omega
26	781.25	vorticity
27	812.50	omega
28	843.75	vorticity
29	875.00	omega
30	906.25	vorticity
31	937.50	omega
32	968.75	vorticity
33	1,000	

$$C_{i+2}-\left[2+4\sigma_i\left(\frac{k\Delta p}{f_0}\right)^2\right] C_i+C_{i-2}=4k\Delta p\left(\frac{k}{f_0}\right)^2 g(U_{i+1}-U_{i-1})B_i-\frac{2\beta k\Delta pg}{f_0^2}(B_{i+1}-B_{i-1}) \tag{25a}$$

and

$$D_{i+2}-\left[2+4\sigma_i\left(\frac{k\Delta p}{f_0}\right)^2\right] D_i+D_{i-2}=-4k\Delta p\left(\frac{k}{f_0}\right)^2 g(U_{i+1}-U_{i-1})A_i+\frac{2\beta k\Delta pg}{f_0^2}(A_{i+1}-A_{i-1}) \tag{25b}$$

$$i=3, 5, 7, \ldots, 31$$

The pressure increment, Δp, is 31.25 mb. At the even grid points,

$$C_i=\frac{C_{i+1}+C_{i-1}}{2},\quad D_i=\frac{D_{i+1}+D_{i-1}}{2} \tag{26}$$

$$i=2, 4, 6, \ldots, 32$$

Boundary conditions are $C_1=C_{33}=D_1=D_{33}=0$. The prognostic equations, (20) and (21), are applied at the even grid points in the approximate forms

$$\frac{\partial A_i}{\partial t}=kC_{Ri}B_i-\left(\frac{f_0}{k}\right)^2 g^{-1}\left(\frac{C_{i+1}-C_{i-1}}{2\Delta p}\right)\equiv\Gamma_i \tag{27a}$$

$$\frac{\partial B_i}{\partial t}=-kC_{Ri}A_i-\left(\frac{f_0}{k}\right)^2 g^{-1}\left(\frac{D_{i+1}-D_{i-1}}{2\Delta p}\right)\equiv\Gamma_i' \tag{27b}$$

$$i=2, 4, 6, \ldots, 32$$

At the odd grid points

$$A_i=\frac{A_{i+1}+A_{i-1}}{2}, \quad B_i=\frac{B_{i+1}+B_{i-1}}{2} \tag{28}$$

$$i=3, 5, 7, \ldots, 31$$

Equations (25a) and (25b) may be written

$$-C_{i+2}+\alpha_i C_i-C_{i-2}=\gamma_i \tag{29}$$

$$-D_{i+2}+\alpha_i D_i-D_{i-2}=\gamma_i' \tag{30}$$

$$i=3, 5, 7, \ldots, 31$$

In (29) and (30)

$$\alpha_i=2+\frac{4\sigma_i k^2\Delta p^2}{f_0^2} \tag{31}$$

$$\gamma_i=\frac{2\beta k\Delta pg}{f_0^2}(B_{i+1}-B_{i-1})-\frac{4k^3\Delta pg}{f_0^2}(U_{i+1}-U_{i-1})B_i \tag{32}$$

and

$$\gamma_i'=\frac{4k^3\Delta pg}{f_0^2}(U_{i+1}-U_{i-1})A_i-\frac{2\beta k\Delta pg}{f_0^2}(A_{i+1}-A_{i-1}) \tag{33}$$

Richtmyer [3] gives a simple method for solving the difference equations (29) and (30). To employ this method, we introduce the following definitions,

$$E_i\equiv\frac{1}{\alpha_i-E_{i-2}} \tag{34}$$

$$F_i\equiv\frac{\gamma_i+F_{i-2}}{\alpha_i-E_{i-2}} \tag{35}$$

$$F_i'\equiv\frac{\gamma_i'+F_{i-2}'}{\alpha_i-E_{i-2}} \tag{36}$$

$$i=3, 5, 7, \ldots, 31 \text{ and } E_1\equiv 0,\ F_1\equiv 0,\ F_1'\equiv 0$$

Then, according to Richtmyer [3],

$$C_i=E_iC_{i+2}+F_i \tag{37}$$

$$D_i=E_iD_{i+2}+F_i' \tag{38}$$

where

$$i=31, 29, 27, \ldots, 3 \text{ and } C_1=C_{33}=D_{33}=D_1=0$$

The prognostic equations (27a) and (27b) are solved as follows. We write

$$A_i^{(n+1)}(t+\Delta t)=A_i(t)+\frac{\Delta t}{2}[\Gamma_i(t)+\Gamma_i^{(n)}(t+\Delta t)] \tag{39}$$

$$B_i^{(n+1)}(t+\Delta t)=B_i(t)+\frac{\Delta t}{2}[\Gamma_i'(t)+\Gamma_i'^{(n)}(t+\Delta t)] \tag{40}$$

$$i=2, 4, 6, \ldots, 32. \quad n=1, 2, 3, \ldots .$$

The superscript, n, denotes the nth estimate at time $t+\Delta t$ while symbols without superscripts denote the final estimate of quantities at time t. Δt is the length of the time step. The iterative process defined by (39) and (40) is continued until

$$|A_i^{(n+1)}(t+\Delta t)-A_i^{(n)}(t+\Delta t)|<\tau \tag{41a}$$

and

$$|B_i^{(n+1)}(t+\Delta t)-B_i^{(n)}(t+\Delta t)|<\tau \tag{41b}$$

where τ is a pre-assigned positive tolerance. Guesses are needed for $\Gamma_i^{(1)}(t+\Delta t)$ and $\Gamma_i'^{(1)}(t+\Delta t)$. These are taken as

$$\Gamma_i^{(1)}(t+\Delta t)=\Gamma_i(t) \tag{42a}$$

and

$$\Gamma_i'^{(1)}(t+\Delta t)=\Gamma_i'(t) \tag{42b}$$

Since the coefficients C and D are implicit in Γ and Γ' and since C and D, in turn, are calculated from A and B (by means of equations (31–38)), it is clear that C and D must be recalculated for each new value of the superscript, n.

The iterative process defined by equations (39) and (40) may be thought of as a generalization of Milne's [4] iterative technique for the solution of a single ordinary differential equation. It is also similar to a method recently employed by Veronis [5]. Milne shows (again for the case of a single ordinary differential equation) that a simple analysis may be performed to determine whether or not his iterative process will converge. However, in our case, where we deal with a *system* of ordinary differential equations (especially in view of the fact that the $\Gamma_i^{(n)}$ and $\Gamma_i'^{(n)}$ are dependent on the solutions of (29) and (30)), convergence can be established only by calculation. In the cases attempted, convergence was attained without difficulty (for all computations, $\Delta t=30$ min., $\tau=10^{-3}$ m.).

The technique described above, together with the values of Δt and τ just given, provides extremely accurate solutions to the test cases described below. It is possible that larger time intervals and tolerances would have provided results which would have been sufficiently accurate for the purposes of the author's previous paper. Indeed, it is possible that a simpler method of time integration would have sufficed. However, the author is inclined to begin with methods that provide, perhaps, a greater degree of accuracy than is required and then to adopt less accurate techniques only if the cost of the more accurate result is unreasonable. Following this philosophy, the technique described above, with the tolerance and time interval given, was coded first. Since this program provided results which were more than adequate at a moderate cost (running time for a 12-hr. solution was about 4 min. on the relatively slow and inexpensive G.E. 225 computer at the National Hurricane Research Laboratory), no modifications to the original program were made.

Admittedly, the exclusion of additional calculations leaves unanswered questions with regard to how closely our solutions could have been reproduced by simpler techniques using larger tolerances and coarser space and time meshes. However, investigation of these problems at a time when we already had an accurate and economical program would have delayed the computations for which the program was written (those reported in [1]) and might have been more expensive than simply proceeding, as we did, with the original program.

It may be argued that, since the physical model is rather crude, little is to be gained by obtaining numerical solutions of more than moderate accuracy. However, when crude numerical techniques are employed, we frequently find it difficult to determine which aspects of the solution are a result of the physical assumptions and which are due to numerical approximations. For this reason, the author is inclined to adopt numerical techniques which are as accurate as can be justified by the economics of the situation and the importance of the physical problem.

The results of the test calculations are presented in the following sections.

4. THE ROSSBY WAVE CASE

When U and σ are constants ($\sigma>0$), equations (20)–(23) may be written

$$\frac{\partial A}{\partial t}=kC_RB-\frac{f_0^2}{k^2g}\frac{\partial C}{\partial p} \tag{43}$$

$$\frac{\partial B}{\partial t}=-kC_RA-\frac{f_0^2}{k^2g}\frac{\partial D}{\partial p} \tag{44}$$

$$\frac{\partial^2 C}{\partial p^2}-\frac{k^2}{f_0^2}\sigma C=-\frac{\beta kg}{f_0^2}\frac{\partial B}{\partial p} \tag{45}$$

$$\frac{\partial^2 D}{\partial p^2}-\frac{k^2}{f_0^2}\sigma D=\frac{\beta kg}{f_0^2}\frac{\partial A}{\partial p} \tag{46}$$

It is noted that C_R (by equation (24)) is also a constant for this problem. If,

$$A(0,p)=0,\ B(0,p)=B_0=\text{a constant}, \tag{47}$$

then equations (45) and (46), together with the boundary conditions on C and D, can be satisfied only by the trivial solutions

$$C(0,p)=D(0,p)=0 \tag{48}$$

From (43), (44), (47), and (48), we find that $\partial A/\partial t$ and $\partial B/\partial t$ at $t=0$ are independent of pressure. If we now differentiate (45) and (46) with respect to time and note that at $t=0$ $\partial^2 B/\partial p\partial t$ and $\partial^2 A/\partial p\partial t$ are zero, we find that the initial values of $\partial C/\partial t$ and $\partial D/\partial t$ are zero. This may in turn be used to show that the initial values of $\partial^2 A/\partial t^2$ and $\partial^2 B/\partial t^2$ are independent of pressure which may then be used to show that the initial values of $\partial^2 C/\partial t^2$ and $\partial^2 D/\partial t^2$ are zero. In the same way, it may be shown that all time derivatives of C, D, $\partial A/\partial p$, and $\partial B/\partial p$ are initially zero. Hence, the motion must be isobaric and invariant with pressure for all time. Equations (43) and (44), therefore, reduce to

$$\partial A/\partial t=kC_RB \tag{49a}$$

and

$$\partial B/\partial t=-kC_RA \tag{49b}$$

In view of the initial conditions, the solution to the problem is

$$A=B_0\sin kC_Rt \tag{50a}$$

$$B=B_0\cos kC_Rt \tag{50b}$$

which shows that the disturbances move with constant amplitude at the Rossby speed.

Numerical solutions were obtained for $L=2{,}000$, 6,000, and 10,000 km. ($t=12$ hr., $\Delta t=30$ min., $\tau=10^{-3}$ m., $f_0=10^{-4}$ sec.$^{-1}$, $\beta=16\times10^{-12}$ m.$^{-1}$ sec.$^{-1}$, $\sigma=0.5$ mts units, $U=\pm10$ m. sec.$^{-1}$, $B_0=97.4$ m.). Table 2 gives the results for the $U=10$ m. sec.$^{-1}$ case (values obtained from the numerical solution were identical at all pressure levels). The amplitudes are accurate to within three significant figures and have a percentage error (to the nearest whole percent) of zero. The phase angles, given to the nearest whole degree, are exact at $L=2{,}000$ and 6,000 km. but are in error by 1° at $L=10{,}000$ km. In the $U=-10$ m. sec.$^{-1}$ case (table 3), the amplitudes are again exact to three significant figures and to the nearest whole percent. The phase angles are exact to the nearest whole degree.

5. ADVECTIVE MODEL WITH BAROTROPIC BASE STATE

If σ is zero and U is constant, equations (20–23) become

$$\frac{\partial A}{\partial t}=kC_RB-\frac{f_0^2}{k^2g}\frac{\partial C}{\partial p} \tag{51}$$

$$\frac{\partial B}{\partial t}=-kC_RA-\frac{f_0^2}{k^2g}\frac{\partial D}{\partial p} \tag{52}$$

$$\frac{\partial^2 C}{\partial p^2}=-\frac{\beta kg}{f_0^2}\frac{\partial B}{\partial p} \tag{53}$$

and

$$\frac{\partial^2 D}{\partial p^2}=\frac{\beta kg}{f_0^2}\frac{\partial A}{\partial p} \tag{54}$$

If we eliminate C between equations (51) and (53) and also eliminate D between equations (54) and (52), we obtain

$$\frac{\partial^2 A}{\partial p\partial t}=kU\frac{\partial B}{\partial p} \tag{55}$$

and

$$\frac{\partial^2 B}{\partial p\partial t}=-kU\frac{\partial A}{\partial p} \tag{56}$$

If we now eliminate B between equations (55) and (56), we find

$$\left(\frac{\partial^2}{\partial t^2}+k^2U^2\right)\frac{\partial A}{\partial p}=0 \tag{57}$$

The general solution to equation (57) is

TABLE 2.—*Comparison between the closed solution (equations (50a) and (50b)) and the numerical solution for the Rossby wave case (U=a constant, σ=a constant, $A(0, p)$=a constant=0, $B(0, p)$=a constant=B_0). Values of the parameters are, $B_0=97.4$ m., $\beta=16\times10^{-12}$ sec.$^{-1}$ m.$^{-1}$, $f_0=10^{-4}$ sec.$^{-1}$, $\sigma=0.5$ mts units, $U=10$ m. sec.$^{-1}$, $t=12$ hr., $\Delta t=30$ min., $\tau=10^{-3}$ m. Amplitudes (R) are in units of meters and are given to three significant figures. Phase angles (δ) are given to the nearest whole degree. Subscript N denotes properties of the numerical solution; values without subscripts are from the closed solution.*

L(km)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
2,000	97.4	97.4	0.00	−65	−65	0
6,000	97.4	97.4	0.00	−12	−12	0
10,000	97.4	97.4	0.00	−46	−47	−1

TABLE 3.—*Same as table 2 except $U=-10$ m. sec.$^{-1}$*

L(km)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
2,000	97.4	97.4	0.00	−90	−90	0
6,000	97.4	97.4	0.00	−64	−64	0
10,000	97.4	97.4	0.00	−79	−79	0

$$A=F_1(p)\sin kUt+F_2(p)\cos kUt+G(t) \tag{58}$$

$F_1(p)$, $F_2(p)$ and $G(t)$ are functions of integration. If we take $A(0,p)=0$ we find

$$F_2(p)=0 \tag{59}$$

$$G(0)=0 \tag{60}$$

By use of (58) and (59),

$$A=F_1(p)\sin kUt+G(t) \tag{61}$$

From (55), (59–61)

$$B=F_1(p)\cos kUt+H(t) \tag{62}$$

where $H(t)$ is a new function of integration. From (51), (52) and the boundary conditions on C and D we find

$$\int_0^{p_0}\left(\frac{\partial A}{\partial t}-kC_RB\right)dp=0 \tag{63}$$

$$\int_0^{p_0}\left(\frac{\partial B}{\partial t}-kC_RA\right)dp=0 \tag{64}$$

From (61–64), we obtain

$$\frac{\beta\overline{F}_1}{k}\cos kUt+\frac{dG}{dt}=kC_RH \tag{65}$$

and

$$-\frac{\beta\overline{F}_1}{k}\sin kUt+\frac{dH}{dt}+kC_RG=0 \tag{66}$$

where

$$\overline{F}_1=\frac{1}{p_0}\int_0^{p_0}F_1(p)dp \tag{67}$$

If H is eliminated between (65) and (66), we have

$$\frac{d^2G}{dt^2}+k^2C_R^2G=\beta(C_R+U)\overline{F}_1\sin kUt \tag{68}$$

The general solution to equation (68) is

$$G=K_1\sin kC_Rt+K_2\cos kC_Rt-\overline{F}_1\sin kUt \tag{69}$$

where K_1 and K_2 are constants of integration. Since $A(0, p)=0$, we have, from (60), $G(0)=0$ and, hence, from (69), $K_2=0$ so that equation (69) reduces to

$$G=K_1\sin kC_Rt-\overline{F}_1\sin kUt \tag{70}$$

From (65) and (70), we find

$$H=-\overline{F}_1\cos kUt+K_1\cos kC_Rt \tag{71}$$

By use of the notation, $B(0, p)=B_0$ (a function of p), we find from equations (62) and (71),

$$B_0=F_1(p)-\overline{F}_1+K_1 \tag{72}$$

Vertical integration of (72) yields

$$\overline{B}_0=K_1 \tag{73}$$

From (72) and (73)

$$B_0=F_1(p)-\overline{F}_1+\overline{B}_0 \tag{74}$$

We will limit ourselves to initial conditions such that $\overline{B}_0=0$. Furthermore, we will select $F_1(p)$ such that $\overline{F}_1=0$. Therefore

$$B_0=F_1(p) \tag{75}$$

and the solution is given by

$$A=B_0\sin kUt \tag{76}$$

$$B=B_0\cos kUt \tag{77}$$

(It should be noted that the requirement $\overline{B}_0=0$ eliminates the wave component which moves with the Rossby speed.) Finally, we choose

$$B_0=B_{00}(2p-p_0)/p_0 \tag{78}$$

where B_{00} is a constant.

We take $f_0=10^{-4}$ sec.$^{-1}$, $\beta=16\times10^{-12}$ m.$^{-1}$ sec.$^{-1}$, $t=12$ hr., $U=10$ m. sec.$^{-1}$, $B_{00}=97.4$ m., $\Delta t=30$ min., $\tau=10^{-3}$ m. The solution for L=2,000 km. is given by table 4. Amplitudes obtained from the numerical solution are exact to three significant figures and to the nearest

TABLE 4.—*Comparison between the closed solution (equations (76) and (77)), and the numerical solution for the advective model with constant zonal wind* ($\sigma=0$, $U=$*a constant*, $A(0,\ p)=0$, $B(0,\ p)=B_{00}\ (2p-p_0)/p_0$). *Values of the parameters are,* $L=2000$ *km,* $f_0=10^{-4}$ *sec.*$^{-1}$, $\beta=16\times10^{-12}$ *m.*$^{-1}$ *sec.*$^{-1}$, $U=10$ *m. sec.*$^{-1}$, $B_{00}=97.4$ *m.,* $t=12$ *hr.,* $\Delta t=30$ *min.,* $\tau=10^{-3}$*m. Amplitudes (R) are in units of meters and are given to three significant figures. Phase angles* (δ) *are given to the nearest whole degree. Subscript N denotes properties of the numerical solution; values without subscripts are from the closed solution.*

Pressure (mb.)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	97.4			258		
31.25	91.4	91.4	0.00	258	258	0
62.50	85.3	85.3	0.00	258	258	0
93.75	79.2	79.2	0.00	258	258	0
125.00	73.1	73.1	0.00	258	258	0
156.25	67.0	67.0	0.00	258	258	0
187.50	60.9	60.9	0.00	258	258	0
218.75	54.8	54.8	0.00	258	258	0
250.00	48.7	48.7	0.00	258	258	0
281.25	42.6	42.6	0.00	258	258	0
312.50	36.5	36.5	0.00	258	258	0
343.75	30.4	30.4	0.00	258	258	0
375.00	24.4	24.4	0.00	258	258	0
406.25	18.3	18.3	0.00	258	258	0
437.50	12.2	12.2	0.00	258	258	0
468.75	6.09	6.09	0.00	258	258	0
500.00	0.00	0.00	0.00			
531.25	6.09	6.09	0.00	78	78	0
562.50	12.2	12.2	0.00	78	78	0
593.75	18.3	18.3	0.00	78	78	0
625.00	24.4	24.4	0.00	78	78	0
656.25	30.4	30.4	0.00	78	78	0
687.50	36.5	36.5	0.00	78	78	0
718.75	42.6	42.6	0.00	78	78	0
750.00	48.7	48.7	0.00	78	78	0
781.25	54.8	54.8	0.00	78	78	0
812.50	60.9	60.9	0.00	78	78	0
843.75	67.0	67.0	0.00	78	78	0
875.00	73.1	73.1	0.00	78	78	0
906.25	79.2	79.2	0.00	78	78	0
937.50	85.3	85.3	0.00	78	78	0
968.75	91.4	91.4	0.00	78	78	0
1000.00	97.4			78		

whole percent of error; phase angles are exact to the nearest whole degree. Table 5 shows that the same high degree of accuracy is achieved by the numerical solution in the case with $L=6{,}000$ km. The calculation was repeated with $L=10{,}000$ km. (results not shown) with equally good results.

6. ADVECTIVE MODEL, BAROCLINIC BASE STATE, $\beta=0$

If we make the change of variable

$$p_*=p/p_0 \tag{79}$$

take $\sigma=0$, $\beta=0$ and choose

$$U=U_0(1-p_*) \tag{80}$$

where U_0 is a constant and p_0 is 1,000 mb., equations (20–23) reduce to

$$\frac{\partial A}{\partial t}=kU_0(1-p_*)B-\frac{f_0^2}{k^2p_0g}\frac{\partial C}{\partial p_*} \tag{81}$$

$$\frac{\partial B}{\partial t}=-kU_0(1-p_*)A-\frac{f_0^2}{k^2p_0g}\frac{\partial D}{\partial p_*} \tag{82}$$

$$\frac{\partial^2 C}{\partial p_*^2}=-\frac{2k^3p_0U_0g}{f_0^2}B \tag{83}$$

TABLE 5.—*Same as table 4 except* $L=6000$ *km.*

Pressure (mb)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	97.4			206		
31.25	91.4	91.4	0.00	206	206	0
62.50	85.3	85.3	0.00	206	206	0
93.75	79.2	79.2	0.00	206	206	0
125.00	73.1	73.1	0.00	206	206	0
156.25	67.0	67.0	0.00	206	206	0
187.50	60.9	60.9	0.00	206	206	0
218.75	54.8	54.8	0.00	206	206	0
250.00	48.7	48.7	0.00	206	206	0
281.25	42.6	42.6	0.00	206	206	0
312.50	36.5	36.5	0.00	206	206	0
343.75	30.4	30.4	0.00	206	206	0
375.00	24.4	24.4	0.00	206	206	0
406.25	18.3	18.3	0.00	206	206	0
437.50	12.2	12.2	0.00	206	206	0
468.75	6.09	6.09	0.00	206	206	0
500.00	0.00	0.00	0.00			
531.25	6.09	6.09	0.00	26	26	0
562.50	12.2	12.2	0.00	26	26	0
593.75	18.3	18.3	0.00	26	26	0
625.00	24.4	24.4	0.00	26	26	0
656.25	30.4	30.4	0.00	26	26	0
687.50	36.5	36.5	0.00	26	26	0
718.75	42.6	42.6	0.00	26	26	0
750.00	48.7	48.7	0.00	26	26	0
781.25	54.8	54.8	0.00	26	26	0
812.50	60.9	60.9	0.00	26	26	0
843.75	67.0	67.0	0.00	26	26	0
875.00	73.1	73.1	0.00	26	26	0
906.25	79.2	79.2	0.00	26	26	0
937.50	85.3	85.3	0.00	26	26	0
968.75	91.4	91.4	0.00	26	26	0
1000.00	97.4			26		

and

$$\frac{\partial^2 D}{\partial p_*^2}=\frac{2k^3p_0U_0g}{f_0^2}A \tag{84}$$

We eliminate C between equations (81) and (83) and eliminate D between (82) and (84). This gives

$$\frac{\partial^2 A}{\partial p_*\partial t}=kU_0B+kU_0(1-p_*)\frac{\partial B}{\partial p_*} \tag{85}$$

and

$$\frac{\partial^2 B}{\partial p_*\partial t}=-kU_0A-kU_0(1-p_*)\frac{\partial A}{\partial p_*} \tag{86}$$

We now eliminate A between (85) and (86) to obtain

$$\frac{\partial^4 B}{\partial p_*^2\partial t^2}+k^2U^2\frac{\partial^2 B}{\partial p_*^2}=0 \tag{87}$$

where $U=U_0(1-p_*)$. From (87),

$$\frac{\partial^2 B}{\partial p_*^2}=F_1(p_*)\sin kUt+F_2(p_*)\cos kUt \tag{88}$$

$F_1(p_*)$ and $F_2(p_*)$ are functions of integration. If the initial conditions are

$$A(0,p_*)=0 \tag{89a}$$

and

$$B(0,p_*)=B_0=\text{a constant} \tag{89b}$$

then $F_2(p_*)=0$. Differentiation of (86) by p_* gives

$$\frac{\partial^2 A}{\partial p_*^2}=-\frac{1}{kU}\frac{\partial^3 B}{\partial p_*^2\partial t}=-F_1(p_*)\cos kUt \tag{90}$$

However, because of (89a), (90) gives $F_1(p_*)=0$. Therefore,

$$\partial^2 A/\partial p_*^2=0 \tag{91}$$

and

$$\partial^2 B/\partial p_*^2=0 \tag{92}$$

which are valid for all time. Direct integration of equations (91) and (92) gives

$$A=S(t)p_*+M(t) \tag{93}$$

and

$$B=G(t)p_*+H(t) \tag{94}$$

where $S(t)$, $M(t)$, $G(t)$ and $H(t)$ are functions of integration. Substitution of (93) and (94) into (85) and (86) gives

$$dS/dt=kU_0G+kU_0H \tag{95}$$

and

$$dG/dt=-kU_0S-kU_0M \tag{96}$$

From (81), (82), (95) and (96), we obtain

$$\frac{f_0^2}{k^2p_0g}\frac{\partial C}{\partial p_*}=\left(kU_0H-\frac{dM}{dt}\right)+\left(kU_0G-kU_0H-\frac{dS}{dt}\right)p_*-kU_0Gp_*^2 \tag{97}$$

and

$$\frac{f_0^2}{k^2p_0g}\frac{\partial D}{\partial p_*}=-\left(kU_0M+\frac{dH}{dt}\right)+\left(kU_0M-kU_0S-\frac{dG}{dt}\right)p_*+kU_0Sp_*^2 \tag{98}$$

Since C and D both vanish at $p_*=0$ and $p_*=1$, equations (97) and (98) may be integrated over the depth of the atmosphere to yield

$$\frac{1}{2}kU_0H+\frac{1}{6}kU_0G-\frac{dM}{dt}-\frac{1}{2}\frac{dS}{dt}=0 \tag{99}$$

and

$$\frac{1}{2}kU_0M+\frac{1}{6}kU_0S+\frac{dH}{dt}+\frac{1}{2}\frac{dG}{dt}=0 \tag{100}$$

Equations (95), (96), (99), and (100) determine G, H, S, and M to within arbitrary constants. These solutions may then be substituted into (93) and (94) and the initial conditions (equations (89a) and (89b)) applied to determine the arbitrary constants. The procedure, though simple, is long and tedious and, for the sake of space, is omitted here. The results obtained are

$$A=B_0[\sin\sqrt{3}\nu t\cosh\nu t-\sqrt{3}(1-2p_*)\cos\sqrt{3}\nu t\sinh\nu t], \tag{101}$$

$$B=B_0[\cos\sqrt{3}\nu t\cosh\nu t+\sqrt{3}(1-2p_*)\sin\sqrt{3}\nu t\sinh\nu t] \tag{102}$$

where

$$\nu=kU_0/(2\sqrt{3}) \tag{103}$$

We do not contend that the solution, as given by equations (101) and (102), is representative of the development process in the real atmosphere nor that it is consistent with the assumptions associated with the linearization of the problem. However, the availability of this solution does allow a rather severe test of the numerical technique for cases in which amplification takes place. The numerical results shown below indicate that the numerical solution yields intensification which is entirely consistent with that given by the solution to the differential equations.

Table 6 compares the solutions for L=2,000 km., $f_0=10^{-4}$ sec.$^{-1}$ $\beta=16\times10^{-12}$ m.$^{-1}$ sec.$^{-1}$, $U_0=40$ m. sec.$^{-1}$, $B_0=97.4$ m., $t=12$ hr, $\Delta t=30$ min. and $\tau=10^{-3}$ m. Errors in the amplitude are 3 m. or less and percentage errors are no larger than 1 percent. The phase angles, at three of the grid points, are in error by 1°. At the remaining grid points they are exact to the nearest whole degree. The results for $L=6{,}000$ km. are shown by table 7. Amplitudes obtained from the numerical solution are exact to three significant figures and to the nearest whole percent of error. The phase angles are exact to the nearest whole degree. Table 8 shows that the results with $L=10{,}000$ km. are equally as good.

TABLE 6.—*Comparisons between the closed solution (equations (101) and (102)) and the numerical solution for the advective model with* $\beta=0$ *and with a baroclinic zonal current* ($\sigma=0$, $U=U_0\ (1-p_*)$, $A(0,p_*)=0$, $B\ (0,p_*)=B_0=$ *a constant). Values of the parameters are,* $L=2000$ *km.,* $B_0=97.4$*m.,* $f_0=10^{-4}$, *sec.*$^{-1}$, $U_0=40$*m. sec.*$^{-1}$, $t=12$ *hr.,* $\Delta t=30$ *min.,* $\tau=10^{-3}$*m. Amplitudes (R) are in units of meters and are given to three significant figures. Phase angles (δ) are given to the nearest whole degree. Subscript N denotes properties of the numerical solution; values without subscripts are properties of the closed solution.*

Pressure (mb.)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	457	----------	----------	108	----------	----------
31.25	437	434	−0.01	99	99	0
62.50	417	414	−0.01	101	101	0
93.75	397	395	−0.01	103	103	0
125.00	379	376	−0.01	105	106	1
156.25	360	358	−0.01	108	108	0
187.50	343	341	−0.01	111	111	0
218.75	326	325	0.00	114	114	0
250.00	311	309	−0.01	117	117	0
281.25	296	295	0.00	121	121	0
312.50	283	282	0.00	125	125	0
343.75	272	270	−0.01	129	129	0
375.00	262	260	−0.01	134	134	0
406.25	254	253	0.00	139	139	0
437.50	248	247	0.00	144	144	0
468.75	245	243	−0.01	150	150	0
500.00	244	242	−0.01	155	156	1
531.25	245	243	−0.01	161	161	0
562.50	248	247	0.00	167	167	0
593.75	254	253	0.00	172	172	0
625.00	262	260	−0.01	177	177	0
656.25	272	270	−0.01	182	182	0
687.50	283	282	0.00	186	186	0
718.75	296	295	0.00	190	190	0
750.00	311	309	−0.01	194	194	0
781.25	326	325	0.00	197	197	0
812.50	343	341	−0.01	200	200	0
843.75	360	358	−0.01	203	203	0
875.00	379	376	−0.01	205	206	1
906.25	397	395	−0.01	208	208	0
937.50	417	414	−0.01	210	210	0
968.75	437	434	−0.01	212	212	0
1000.00	457	----------	----------	213	----------	----------

TABLE 7.—*Same as table 6 except L=6000 km.*

Pressure (mb.)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	144			12		
31.25	141	141	0.00	14	14	0
62.50	137	137	0.00	16	16	0
93.75	134	134	0.00	18	18	0
125.00	131	131	0.00	20	20	0
156.25	128	128	0.00	22	22	0
187.50	125	125	0.00	24	24	0
218.75	123	123	0.00	27	27	0
250.00	120	120	0.00	29	29	0
281.25	118	118	0.00	32	32	0
312.50	116	116	0.00	35	35	0
343.75	115	115	0.00	37	37	0
375.00	113	113	0.00	40	40	0
406.25	112	112	0.00	43	43	0
437.50	112	112	0.00	46	46	0
468.75	111	111	0.00	49	49	0
500.00	111	111	0.00	52	52	0
531.25	111	111	0.00	55	55	0
562.50	112	112	0.00	58	58	0
593.75	112	112	0.00	61	61	0
625.00	113	113	0.00	64	64	0
656.25	115	115	0.00	66	66	0
687.50	116	116	0.00	69	69	0
718.75	118	118	0.00	72	72	0
750.00	120	120	0.00	74	74	0
781.25	123	123	0.00	77	77	0
812.50	125	125	0.00	79	79	0
843.75	128	128	0.00	82	82	0
875.00	131	131	0.00	84	84	0
906.25	134	134	0.00	86	86	0
937.50	137	137	0.00	88	88	0
968.75	141	141	0.00	90	90	0
1000.00	144			92		

TABLE 9.—*Same as table 6 except t=48 hr.*

Pressure (mb.)	R $\times 10^2$	R_N $\times 10^2$	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	513			202		
31.25	489	476	−0.03	203	204	1
62.50	465	454	−0.02	205	205	0
93.75	442	431	−0.02	207	207	0
125.00	420	410	−0.02	209	210	1
156.25	399	388	−0.03	212	212	0
187.50	378	368	−0.03	214	215	1
218.75	358	349	−0.03	218	218	0
250.00	339	330	−0.03	221	221	0
281.25	322	313	−0.03	225	225	0
312.50	306	298	−0.03	229	229	0
343.75	291	284	−0.02	233	234	1
375.00	279	272	−0.03	238	239	1
406.25	269	262	−0.03	244	244	0
437.50	262	255	−0.03	250	250	0
468.75	258	251	−0.03	256	256	0
500.00	256	250	−0.02	262	262	0
531.25	258	251	−0.03	268	268	0
562.50	262	255	−0.03	274	274	0
593.75	269	262	−0.03	280	280	0
625.00	279	272	−0.03	286	286	0
656.25	291	284	−0.02	290	291	1
687.50	306	298	−0.03	295	295	0
718.75	322	313	−0.03	299	299	0
750.00	339	330	−0.03	302	303	1
781.25	358	349	−0.03	306	306	0
812.50	378	368	−0.03	309	309	0
843.75	399	388	−0.03	312	312	0
875.00	420	410	−0.02	314	315	1
906.25	442	431	−0.02	316	317	1
937.50	465	454	−0.02	318	319	1
968.75	489	476	−0.03	320	320	0
1000.00	513			322		

TABLE 8.—*Same as table 6 except L=10,000 km.*

Pressure (mb.)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	116			3		
31.25	114	114	0.00	5	5	0
62.50	113	113	0.00	6	6	0
93.75	111	111	0.00	8	8	0
125.00	110	110	0.00	10	10	0
156.25	109	109	0.00	11	11	0
187.50	108	108	0.00	13	13	0
218.75	107	107	0.00	15	15	0
250.00	106	106	0.00	16	16	0
281.25	105	105	0.00	18	18	0
312.50	104	104	0.00	20	20	0
343.75	104	104	0.00	22	22	0
375.00	103	103	0.00	24	24	0
406.25	103	103	0.00	25	25	0
437.50	102	102	0.00	27	27	0
468.75	102	102	0.00	29	29	0
500.00	102	102	0.00	31	31	0
531.25	102	102	0.00	33	33	0
562.50	102	102	0.00	35	35	0
593.75	103	103	0.00	37	37	0
625.00	103	103	0.00	39	39	0
656.25	104	104	0.00	40	40	0
687.50	104	104	0.00	42	42	0
718.75	105	105	0.00	44	44	0
750.00	106	106	0.00	46	46	0
781.25	107	107	0.00	48	48	0
812.50	108	108	0.00	49	49	0
843.75	109	109	0.00	51	51	0
875.00	110	110	0.00	53	53	0
906.25	111	111	0.00	54	54	0
937.50	113	113	0.00	56	56	0
968.75	114	114	0.00	57	57	0
1000.00	116			59		

TABLE 10.—*Same as table 6 except L=6000 km. and t=48 hr.*

Pressure (mb.)	R	R_N	$\frac{R_N-R}{R}$	δ	δ_N	$\delta_N-\delta$
00.00	781			148		
31.25	745	743	−0.00	150	150	0
62.50	710	708	−0.00	151	152	1
93.75	676	674	−0.00	153	154	1
125.00	642	641	−0.00	156	156	0
156.25	610	608	−0.00	158	158	0
187.50	579	577	−0.00	161	161	0
218.75	549	548	−0.00	164	164	0
250.00	522	520	−0.00	167	167	0
281.25	496	494	−0.00	171	171	0
312.50	472	471	−0.00	175	175	0
343.75	451	450	−0.00	180	180	0
375.00	433	432	−0.00	184	185	1
406.25	419	417	−0.00	190	190	0
437.50	408	407	−0.00	195	195	0
468.75	402	400	−0.00	201	201	0
500.00	399	398	−0.00	207	207	0
531.25	402	400	−0.00	213	213	0
562.50	408	407	−0.00	219	219	0
593.75	419	417	−0.00	225	225	0
625.00	433	432	−0.00	230	230	0
656.25	451	450	−0.00	235	235	0
687.50	472	471	−0.00	239	240	1
718.75	496	494	−0.00	244	244	0
750.00	522	520	−0.00	247	247	0
781.25	549	548	−0.00	250	251	0
812.50	579	577	−0.00	254	254	0
843.75	610	608	−0.00	256	257	1
875.00	642	641	−0.00	259	259	0
906.25	676	674	−0.00	261	261	0
937.50	710	708	−0.00	263	263	0
968.75	745	743	−0.00	265	265	0
1000.00	781			266		

As a final demonstration of the accuracy of the numerical technique, calculations were extended to 48 hr. with $\beta=0$, $B_0=97.4$ m., $f_0=10^{-4}$ sec.$^{-1}$, $U_0=40$ m. sec.$^{-1}$, $\Delta t=30$ min. $\tau=10^{-3}$ m. The results, with $L=2{,}000$ km., are given by table 9. It will be noted that although amplitudes (as given by the closed solution) have increased by a factor in excess of 10^2, the numerical solution gives the correct value to within 2–3 percent. Phase angles obtained from the numerical solution are correct to within 1° or less. Table 10 gives the results with $L=6000$ km. Here, the amplitudes obtained from the numerical solution are correct to within 1–2 m.; percentage errors, to the nearest whole percent, are zero. Phase angles are accurate to within 1°.

7. SUMMARY

In a previous paper [1], the author proposed a numerical scheme for obtaining solutions to an initial value problem defined by linearized equations for quasi-geostrophic flow. Tests to determine the accuracy of the scheme were conducted but were not discussed in [1]. Since both the problem and the numerical method appear to be of general interest, the present paper was prepared for the purpose of making the results of these tests available. The tests consisted of applications of the numerical method to three simple models for which it was possible to obtain closed solutions. The numerical solutions were found to be remarkably good.

The first test model was one in which the mean zonal current and the mean static stability were constant in the vertical. The amplitude and phase angle of the geopotential perturbation were initially constant in the vertical. The closed solution gives a disturbance moving at the Rossby speed with constant amplitude and for which the motion is isobaric and invariant with pressure for all time. Numerical solutions were obtained out to 12 hr. with half-hour time steps for wavelengths of 2,000, 6,000, and 10,000 km. The results showed amplitudes which were accurate to three significant figures and phase angles which were accurate to within 1°.

In the second test case, the mean zonal current was again barotropic; the mean static stability was zero. The amplitude of the geopotential perturbation was initially a linear function of pressure (zero at the 500-mb. level). The phase of the initial geopotential perturbation was constant above and below the 500-mb. level with the perturbation in the upper half of the atmosphere being 180° out of phase with that in the lower troposphere. The closed solution gave a wave moving with constant amplitude at the speed of the basic current. Again, numerical solutions were obtained for $L=2{,}000$, 6,000, and 10,000 km. with half-hour time steps out to 12 hr. For all three wavelengths, the numerical calculation gave amplitudes which were exact to the nearest whole degree.

The third model was baroclinic with the basic zonal current being a linear function of pressure; the mean static stability was zero and the beta-parameter was zero. The amplitude and phase angle of the geopotential perturbation were initially constant with pressure. For this case, the closed solution, of course, gave disturbances which increased in amplitude with time. Again, numerical solutions out to 12 hr. with half-hour time steps were obtained for wavelengths of 2,000, 6,000, and 10,000 km. At $L=2{,}000$ km. the amplitude increased by a factor of 2–5 over the 12-hr. period. Amplitudes obtained from the numerical solution were, however, correct to within 2–3 m. and had percentage errors of 1 percent or less. Phase angles at three of the grid points were in error by 1°; at all other grid points, the numerical solution gave phase angles which were exact to the nearest whole degree. At $L=6{,}000$ and 10,000 km., the amplitudes were exact to three significant figures and to the nearest whole percent of error; phase angles were exact to the nearest whole degree.

As a final test, numerical solutions to the baroclinic model were extended to 48 hr. for the cases with $L=2{,}000$ and 6,000 km. At $L=2{,}000$ km. the amplitudes increased by a factor in excess of 10^2 during the 48-hr. period. The numerical solution, however, was accurate to within 2–3 percent. Phase angles were correct to within 1°. At $L=6{,}000$ km. the amplitude increased by a factor of 4–8 over the 48-hr. period. Nevertheless, the numerical solution was accurate to within 1–2 m. and had percentage errors, to the nearest whole percent, of zero. Phase angles given by the numerical solution were in error by 1° at five grid points; at the remaining grid points, the phase angles were exact to the nearest whole degree.

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